Recipes for Degrees of Freedom of Frequency Stability Estimators

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Abstract—**The Allan variance for an averaging time τ can be estimated either from all available phase samples or from a subgrid of samples with spacing τ. This paper gives a set of computational recipes that yield the variance of both estimators, with less than 2% error, for the five power-law components of the classical continuous-time clock noise model.**

I. Introduction

A. Estimators of Allan Variance

Let $X(t)$ be the time difference of two clocks at time t. If the clocks are both derived from oscillators running at nominal frequency f_0, then $X(t) = \phi(t)/(2\pi f_0)$, where $\phi(t)$ is the difference of the phases of the two oscillators. Thus, if scaling is unimportant, the terms "time deviation" and "phase deviation" are often used interchangeably.

Define the second increment $Z(t, \tau)$ by

$$\begin{aligned} Z(t, \tau) &= X(t) - 2X(t + \tau) + X(t + 2\tau) \\ &= \Delta_\tau^2 X(t) \end{aligned}$$

where Δ_τ is the forward difference operator: $\Delta_\tau f(t) = f(t + \tau) - f(t)$ for any function f. Assume that $X(t)$ can be modeled as a random process with wide-sense stationary second increments. This means that for each τ the process $Z(t, \tau)$ is a wide-sense stationary random function of t. The Allan variance, or two-sample variance, is then defined for $\tau > 0$ by

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} \mathcal{E} Z^2(t, \tau)$$

where $\mathcal{E}$ is the operator of mathematical expectation or ensemble average. This is the standard measure of mean-squared stability of the average fractional frequency $\Delta_\tau X(t)/\tau$ [11].

Given N equally spaced samples $X_i = X(i\tau_0)$, $i = 0$ to $N - 1$, and a desired averaging time $\tau = n\tau_0$, where n is an integer satisfying $1 \le n < N/2$, one can form $M = N - 2n$ samples of $Z(t, \tau)$, namely,

$$Z(i\tau_0, \tau) = X_i - 2X_{i+n} + X_{i+2n}, \qquad i = 0 \text{ to } M - 1.$$

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The *maximal-overlap* estimator of $2\tau^2\sigma_y^2(\tau)$ is given by

$$V(M, \tau, \tau_0) = \frac{1}{M} \sum_{i=0}^{M-1} Z^2(i\tau_0, \tau). \tag{1}$$

Although Yoshimura [10] calls this the "continuous sampling method," we wish to reserve the term *continuous overlap* for the limiting case of (1) in which M and τ/τ_0 both tend to infinity with the ratio $M\tau_0/\tau$ tending to a positive value. A more traditional estimator, called the *tau-overlap* estimator, is given by

$$V(m, \tau, \tau) = \frac{1}{m} \sum_{i=0}^{m-1} Z^2(i\tau, \tau) \tag{2}$$

where m is the number of τ-spaced samples of $Z(t, \tau)$. In the above sampling scheme, $m = \text{int}((N - 1)/n) - 1$, where int (x) is the greatest integer less than or equal to x.

Both estimators are positive and unbiased. A measure of quality that can be applied to a positive unbiased estimator V with finite variance is its *degrees of freedom* (d.f.), defined by

$$\text{d.f.} = \frac{2(\mathcal{E} V)^2}{\text{var } V}.$$

Theoretical and Monte Carlo studies [15], [5], [6] on Gaussian clock noise models have shown that the estimator $V = V(M, \tau, \tau_0)$ is *approximately* proportional in distribution to the chi-square random variable with the same d.f. as V. Thus, if it is known that d.f. $= \nu$, then the probability levels of the χ_ν^2 distribution can be used to derive approximate confidence intervals for the unknown value $\mathcal{E} V$ (see [6] for instructions).

B. Motivation for the Present Work

For the tau-overlap estimator, Lesage and Audoin [1]–[4] and Yoshimura [5] computed d.f. for the five power-law components of the classical continuous-time clock noise model [11], a description of which is given in Section II. This work is essentially complete. For the maximal-overlap estimator, Howe *et al.* [6] gave a set of formulas for d.f. for the five clock noise components. Some of these formulas are empirical and based partly on simulations. Recently, using exact theory, Yoshimura [10] gave a set of graphs giving maximal-overlap d.f. for each clock noise component as a function of M and n for $1 \le M \le 100$ and $1 \le n \le 50$ to 100, and also gave graphical comparisons of the performance of the maximal-overlap and tau-overlap estimators as functions of N and n. Although Yoshimura's results appear to be accurate, one might wish to have programmable formulas for them, or to obtain values of d.f. for M or n outside these ranges. Also, for the flicker-phase component, d.f. depends on a time-bandwidth product whose value is fixed in Yoshimura's graphs. Thus, the purpose of the present work is to give a set of recipes that compute the theoretical d.f. of the maximal-overlap estimator, with the d.f. of the tau-overlap estimator included as a special case. Whenever exact closed-form expressions could be achieved, they were used. Otherwise, approximation formulas whose maximal error is 2% for all $M \geq 1$ and $n \geq 1$ were constructed. The resulting set of formulas was converted to pseudocode, which, in turn, was coded in Fortran and Basic.

The exact value of d.f. can easily be computed from a sum of M terms (see (5), (6), and the formulas for $r(x, \tau)$ below). Nevertheless, the recipes may save computer time and give greater insight into the roles of the variables. In particular, they can be used for evaluating limiting situations, for example, the continuous-overlap limit mentioned above, or the large-sample limit in which n is fixed and M becomes large.

II. Method of Computation

We began by assuming that the time deviation process $X(t)$ has stationary second increments $Z(t, \tau)$. It is now necessary to assume further that $Z(t, \tau)$ is a mean-zero Gaussian process, and is ergodic in the sense that the time average of $Z(t, \tau)$ over $0 \leq t \leq T$ tends to zero in the mean-square sense as T tends to infinity.

The mean-zero assumption is crucial. It says that the relative long-term frequency drift rate D of the two clocks is zero, or that it has previously been measured and the appropriate term $Dt^2/2$ subtracted from $X(t)$. The results given here are invalid for all τ large enough that linear frequency drift dominates the Allan variance. Moreover, if one uses the current data to estimate drift rate in order to subtract it from the data, estimates of residual Allan variance can have a large negative bias if τ is not much less than the data duration [7]. This is caused by the high-pass filtering effect of the subtraction [13].

A. Expression for Degrees of Freedom

For fixed τ, let $Z_i = Z(i\tau_0, \tau)$, and $V = (1/M) \Sigma_{i=0}^{M-1} Z_i^2$, the maximal-overlap estimator defined by (1). We want to calculate the variance of V, namely,

$$\operatorname{var} V = \mathcal{E} V^2 - (\mathcal{E} V)^2$$
$$= \frac{1}{M^2} \sum_i \sum_j [\mathcal{E}(Z_i^2 Z_j^2) - (\mathcal{E} Z_i^2)(\mathcal{E} Z_j^2)].$$

It can be shown that the expectation of the product of any four mean-zero, jointly-Gaussian random variables T_1, T_2, T_3, T_4 satisfies

$$\mathcal{E}(T_1T_2T_3T_4) = \mathcal{E}(T_1T_2)\mathcal{E}(T_3T_4) + \mathcal{E}(T_1T_3)\mathcal{E}(T_2T_4) + \mathcal{E}(T_1T_4)\mathcal{E}(T_2T_3).$$

Setting $T_1 = T_2 = Z_i$, $T_3 = T_4 = Z_j$ leads to

$$\operatorname{var} V = \frac{1}{M^2} \sum_i \sum_j 2[\mathcal{E}(Z_i Z_j)]^2.$$

For each fixed τ, let $R(t, \tau)$ be the autocovariance function of the stationary process $Z(t, \tau)$. Then

$$R(t - u, \tau) = \mathcal{E}(Z(t, \tau)Z(u, \tau)).$$

In terms of this function, the mean and variance of V can be written

$$\mathcal{E} V = R(0, \tau),$$
$$\operatorname{var} V = \frac{2}{M^2} \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} R^2((i - j)\tau_0, \tau)$$

which, after a change of variable in the double sum, becomes

$$\operatorname{var} V = \frac{2}{M}\left[R^2(0, \tau) + 2 \sum_{k=1}^{M-1} (1 - k/M) R^2(k\tau_0, \tau)\right]. \tag{3}$$

It is convenient to introduce a scaled version of $R(t, \tau)$. Let

$$r(x, \tau) = R(\tau x, \tau) \tag{4}$$

and, recalling that $\tau = n\tau_0$, let

$$x_k = k/n, \quad p = M/n.$$

Then the degrees of freedom, d.f. $= 2(\mathcal{E} V)^2/\operatorname{var} V$, is given by

$$\text{d.f.} = \frac{M}{\text{Fct}} \tag{5}$$

where

$$\text{Fct} = \frac{1}{r^2(0, \tau)}\left[r^2(0, \tau) + 2 \sum_{k=1}^{M-1} (1 - x_k/p) r^2(x_k, \tau)\right] \tag{6}$$

(Yoshimura's terminology). Notice that Fct$/n$ is the trapezoidal approximation to the integral

$$\text{Fac} = \frac{2}{r^2(0, \tau)} \int_0^p (1 - x/p) r^2(x, \tau)\, dx. \tag{7}$$

Some of the recipes compute Fac and apply correction terms to get Fct$/n$. This is convenient because, for some of the power-law noise models treated below, $r(x, \tau)/r(0, \tau)$ is a continuous function that depends only on x, not τ. In this case, Fac depends only on p, and is the continuous-overlap limit of Fct$/n$.

B. Autocovariance Functions

The autocovariance function $R(t, \tau)$ of the second increments $Z(t, \tau)$ must now be obtained for the components of the clock noise model. Lesage and Audoin [1] obtained $R(t, \tau)$ in terms of an integral over the frequency domain, although they started from a time-domain formula that is formally identical to (8) below. Yoshimura [5], [10] was able to express $R(t, \tau)$ in terms of the K-sample variance $\sigma_y^2(K, \tau)$, expressions for which are known [14]. Notwithstanding these derivations, the author feels that the method of the generalized autocovariance function [7], [8] deserves further attention because of its simplicity and its applicability to a variety of covariance computations. This theory shows that for any real process $X(t)$ with wide-sense stationary nth increments there is a (non-unique) continuous function, call it $R_X(t)$, such that

$$\mathcal{E}[\Delta_\tau^n X(t)\, \Delta_\tau^n X(u)] = \delta_\tau^{2n} R_X(t - u)$$

where δ_τ^{2n} is the central difference operator of order $2n$. In particular, for $n = 2$ we have

$$\begin{aligned} R(t, \tau) &= \delta_\tau^4 R_X(t) \\ &= 6R_X(t) - 4R_X(t - \tau) - 4R_X(t + \tau) \\ &\quad + R_X(t - 2\tau) + R_X(t + 2\tau). \end{aligned} \tag{8}$$

If X is stationary, like white phase noise, then $R_X(t)$ is the usual autocovariance function of X. If X has stationary first increments but is not stationary, like flicker phase noise and white frequency noise, then $R_X(t)$ equals $-(1/2)\mathcal{E}[X(t) - X(0)]^2$ plus an arbitrary constant. If X has stationary second increments but does not have stationary first increments, like flicker frequency noise and random walk frequency noise, then the author does not know any time-domain second-moment interpretation of $R_X(t)$, which is derived by performing a certain generalized Fourier transform operation on the spectral density of X [8]. Because one always applies fourth-order difference operators to $R_X(t)$, one may add to it an arbitrary polynomial of t of degree ≤ 3. As noted above, (8) gives rigorous meaning to the terms of (17) of Lesage and Audoin [1].

The scaled autocovariance function $r(x, \tau)$ defined by (4) is given by

$$r(x, \tau) = \delta_1^4 f_\tau(x)$$

where $f_\tau(x) = R_X(\tau x)$. For power-law noises, it often turns out that the scaled autocorrelation function $r(x, \tau)/r(0, \tau)$ is a function of x only. This is a self-similarity property.

Let us now give $R_X(t)$ and $r(x, \tau)$ for the components of the classical clock noise signal. The one-sided spectral density of $X(t)$ is denoted by $S_X(t)$. Although the conventional multiplicative constants are irrelevant to the present work, they are given here for the sake of completeness. In the $r(x, \tau)$ formulas, however, the constants are reset to whatever seems convenient. Moreover, except for flicker phase noise, the indicated dependence on τ can be dropped.

White Phase:

$$\begin{aligned} S_X(f) &= \frac{h_2}{4\pi^2}, \qquad f < f_h \\ &= 0, \qquad f > f_h. \end{aligned}$$

$$R_X(0) = \frac{h_2 f_h}{4\pi^2} = \operatorname{var} X,$$

$$R_X(t) = o(1), \qquad \omega_h|t| \gg 1.$$

where $\omega_h = 2\pi f_h$, and $o(1)$ is an error term that tends to zero as t tends to infinity. For $r(x)$, we have

$$r(0) = 6, \quad r(\pm 1) = -4, \quad r(\pm 2) = 1$$

and $r(x)$ is small if the distance of the point $|x|$ from the set $\{0, 1, 2\}$ is much greater than $1/(\omega_h \tau)$. This condition is satisfied for x_k in (6) if k is not equal to n or $2n$, and $\omega_h \tau_0 \gg 1$ (in other words, if τ_0 is much greater than the decorrelation time of $X(t)$).

Flicker Phase:

$$\begin{aligned} S_X(f) &= \frac{h_1}{4\pi^2} f^{-1}, \qquad f < f_h \\ &= 0, \qquad f > f_h. \end{aligned}$$

$$R_X(t) = \frac{h_1}{4\pi^2} \cdot \begin{cases} \gamma, & t = 0, \\ -\log(\omega_h|t|) + o(1), & \omega_h|t| \gg 1, \end{cases}$$

where γ is Euler's constant, $0.5772\cdots$. Before computing $r(x, \tau)$, it is convenient to add $\log(\omega_h \tau)$ to $R_X(t)$; then the modified $R_X(\tau x)$ is $\gamma + \log(\omega_h \tau)$ for $x = 0$, and $-\log|x| + o(1)$ for $|x| \gg 1/(\omega_h \tau)$. Setting

$$L = \gamma + \log(\omega_h \tau)$$

and applying the δ_1^4 operator gives

$$\begin{aligned} r(0, \tau) &= 6L - 2\log 2, \\ r(1, \tau) &= -4L + 4\log 2 - \log 3, \\ r(2, \tau) &= L - 8\log 2 + 4\log 3, \\ r(x, \tau) &= -\delta_1^4 \log|x| \end{aligned}$$

if $|x|$ stays away from $\{0, 1, 2\}$ as with white phase noise.

White Frequency:

$$S_X(f) = \frac{h_0}{4\pi^2} f^{-2},$$

$$R_X(t) = -\frac{h_0}{4}|t|,$$

$$\begin{aligned} &r(0) = 2, \quad r(\pm 1) = -1, \\ &r(x) = 0, \qquad |x| \geq 2, \end{aligned}$$

and $r(x)$ is linear for $0 \leq |x| \leq 1$ and $1 \leq |x| \leq 2$.

Flicker Frequency:

$$S_X(f) = \frac{h_{-1}}{4\pi^2} f^{-3},$$

$$R_X(t) = \frac{h_{-1}}{2} t^2 \log |t|$$

(and $R_X(0) = 0$). Since $R_X(\tau x) = \text{const} \cdot \tau^2 x^2 (\log \tau + \log |x|)$, which effectively is $x^2 \log |x|$, we have

$$r(x) = \delta_1^4(x^2 \log |x|).$$

Random Walk Frequency:

$$S_X(f) = \frac{h_{-2}}{4\pi^2} f^{-4},$$

$$R_X(t) = \frac{h_{-2}\pi^2}{6} |t|^3,$$

and $R_X(\tau x)$ effectively is $|x|^3$. Applying δ_1^4 gives

$$\begin{aligned} r(x) &= 4 - 6x^2 + 3x^3, \quad 0 \le x \le 1, \\ &= (2 - x)^3, \quad 1 \le x \le 2, \\ &= 0, \quad x \ge 2, \end{aligned}$$

and $r(-x) = r(x)$.

Fig. 1 shows the qualitative differences among the autocorrelation functions $r(x)/r(0)$ for the five noise types, where $\omega_h \tau = 100$ for flicker phase noise.

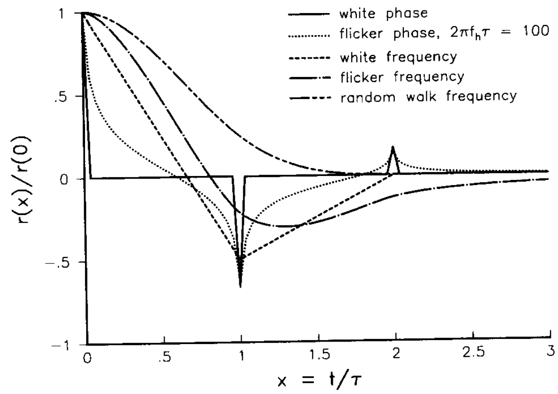

Fig. 1. Autocorrelation functions of the second τ-increment of phase.

III. Overview of Recipes

Each of the five clock-noise power law processes has its own recipe, whose aim is the exact or approximate computation of the d.f. of the Allan variance estimator (1). The exact d.f. is given by (5) and (6), where $r(x, \tau)$, the scaled autocovariance function of the second increments, is given in the previous section for each noise type.

The inputs of each recipe are $n = \tau/\tau_0$, and M = number of second τ-increment samples overlapped by τ_0. For white phase and flicker phase, the time-bandwidth assumption $2\pi f_h \tau_0 >> 1$ is required, where f_h is the high-frequency cutoff of the phase spectral density. For flicker phase, $2\pi f_h \tau_0$ must also be supplied as an input parameter.

The output of the recipe is d.f., the degrees of freedom of the maximal-overlap estimator. If $M = 1$ then d.f. = 1.

For the tau-overlap estimator, one applies the above inputs and time-bandwidth assumption after setting $\tau_0 = \tau$ (so $n = 1$) and $M = m$, the number of second τ-increment samples overlapped by τ.

The recipes consist mainly of piecewise-defined formulas. Pseudocode for them is given in Fig. 2. On request, the author will supply a more readable copy of the pseudocode with explanatory notes, plus a disk containing Fortran 77 and Basic code for recipe subroutines and demo driver programs.

IV. Numerical Results

Table I gives the outputs of these recipes for the same values of N and n as Stein's Table 12-5 [9], which was based partly on the formulas of Howe *et al.* [6] (note that Stein's m corresponds to our n). Several rows have been added to give d.f. values for certain small values of M/n so that users of the recipes can check their implementations. Most of the d.f. results of the two tables agree within a few percent. For white phase, there is complete agreement, except that the top left entry of Stein's table may be a misprint. For flicker phase, the differences become significant (almost a factor of 2) for the larger τ. There may be a difference between the mathematical models; while no value of $2\pi f_h \tau_0$ is specified in Stein's table, Table I arbitrarily assumes a value of 10. For white frequency, the tables differ by less than 2%. For flicker frequency and random walk frequency, the differences are less than 8%. In the case of random walk frequency, the differences appear to be caused, at least partly, by the assumption of a discrete-time random walk model, in which the second τ_0-increments of phase are uncorrelated. In the continuous-time model used here, successive samples of these increments have correlation 1/4.

In summary, except for the flicker phase model, the d.f. values produced by the recipes do not differ radically from values computed in the past. The main contribution of these recipes is the refinement of the values to a 2% level of agreement with theory for all N and τ.

V. Concluding Remarks

The author has constructed a set of recipes for computing the degrees of freedom of the maximal-overlap and tau-overlap estimators of Allan variance of the five components of the classical clock noise model. Where approximations have been used, the errors of the results are less than 1% except for flicker phase noise, in which case some of the errors reach 2%. The results agree with values read from Fig. 2 of Yoshimura [10]; thus the recipes represent those graphs and extend them to any number of samples N, averaging time τ, and cutoff frequency f_h subject to $2\pi f_h \tau_0 >> 1$.

Define $x_+ = \max(x, 0)$.
If $M = 1$ then d.f. = 1. Therefore, assume $M \geq 2$.
Let $p = M/n$.

WHITE PHASE

$$\mathrm{Fct} = 1 + \frac{8}{9}(1 - 1/p)_+ + \frac{1}{18}(1 - 2/p)_+$$

$$\mathrm{d.f.} = \frac{M}{\mathrm{Fct}}$$

FLICKER PHASE

$L = \gamma + \log(2\pi f_h n\tau_0)$! $\gamma = 0.5772\ldots$
$k_0 = 6,\ k_1 = -4,\ k_2 = 1$
$a_0 = 2\log 2,\ a_1 = -4\log 2 + \log 3,\ a_2 = 8\log 2 - 4\log 3$
For i = 0 to 2
  $r_i = k_i L - a_i$
  $q_i = k_i \log n - a_i$
Next i
If $n = 1$ then

$$\mathrm{Fct} = 1 + \frac{2}{r_0^2}\left[r_1^2(1 - 1/M) + r_2^2(1 - 2/M)_+\right]$$

$$\mathrm{d.f.} = \frac{M}{\mathrm{Fct}}$$

Else
  Evaluate $\Phi(p)$ by the recipe below.
  ! $\Phi(p) = r_0^2\,\mathrm{Fac}$

$$\mathrm{Denom} = \Phi(p) + \frac{1}{n}\left\{r_0^2 - (q_0 + 2k_0)^2 + \frac{k_0}{M}(q_0 + k_0/2)\right\} + \frac{2}{n}(1 - 1/p)_+\left[r_1^2 - (q_1 + 2k_1)^2\right] + \frac{2}{n}(1 - 2/p)_+\left[r_2^2 - (q_2 + 2k_2)^2\right]$$

$$\mathrm{d.f.} = \frac{p r_0^2}{\mathrm{Denom}}$$

Endif

<u>$\Phi(p)$ Recipe</u>

If $p \leq .5$ then

$$\Phi(p) = p[36(\log p)^2 - 91.36\log p + 102.97] + p^3(7.36\log p - 2.82)$$

Else if $p \leq 1$ then

$$\Phi(p) = 39.59 + 187.75\,p - 216.88\,p^2 + 92.08\,p^3$$

Else if $p \leq 2$ then

$$\Phi(p) = 77.513 - 78.144\,p + 183.382\,p^2 - 97.153\,p^3 + 16.794\,p^4$$

Else

$$\Phi(p) = 20\pi^2 - \frac{102.64}{p}$$

Endif

WHITE FREQUENCY

If $n = 1$ then

$$\mathrm{Fctvn} = \frac{3}{2} - \frac{1}{2M} \qquad \text{! Fctvn = Fct/n}$$

Else
  If $p \leq 1$ then

$$\mathrm{Fctvn} = p\left(1 - p + \frac{3}{8}p^2\right) + \frac{1}{n^2}\left(1 - \frac{3}{8}p\right)$$

  Else if $p \leq 2$ then

$$\mathrm{Fctvn} = \frac{2}{3} - \frac{1}{3p} + \frac{(2 - p)^4}{24p} + \frac{1}{n^2}\left(1 - \frac{1}{24}p - \frac{1}{3p}\right)$$

  Else

$$\mathrm{Fctvn} = \frac{2}{3} - \frac{1}{3p} + \frac{1}{n^2}\left(\frac{5}{6} - \frac{1}{6p}\right)$$

  Endif
Endif

$$\mathrm{d.f.} = \frac{p}{\mathrm{Fctvn}}$$

FLICKER FREQUENCY

If $n = 1$ then

$$\mathrm{Fctvn} = 1.1354 - \frac{0.1879}{M} \qquad \text{! not for } M = 1$$

Else if $n = 2$ then

$$\mathrm{Fctvn} = 0.7743 - \frac{0.1607}{p} + 0.0799\left(1 - \frac{3}{2p}\right)_+ + 0.0251\left(1 - \frac{2}{p}\right)_+$$

Else
  Define

$$r(p) = 6p^2\log p - 4|p-1|^2\log|p-1| - 4(p+1)^2\log(p+1) + |p-2|^2\log|p-2| + (p+2)^2\log(p+2) \qquad \text{! } 0^2\log 0 = 0$$

  Evaluate Fac by the recipe below.

$$\mathrm{Fctvn} = \mathrm{Fac} + \frac{1.3}{6n^2 p}\left[1 - \frac{r^2(p)}{r^2(0)}\right]$$

Endif

$$\mathrm{d.f.} = \frac{p}{\mathrm{Fctvn}}$$

<u>Fac Recipe</u>

If $p < .5$ then

$$\mathrm{Fac} = p + \frac{p^3}{4\log 2}(\log p - 1.58)$$

Else if $p < 2$ then

$$\mathrm{Fac} = -0.0581 + 1.4547\,p - 1.3602\,p^2 + 0.6176\,p^3 - 0.1054\,p^4$$

Else

$$\mathrm{Fac} = \frac{\pi^2}{24(\log 2)^2} - \frac{0.3911}{p} + \frac{0.02}{p^2}$$

Endif

RANDOM WALK FREQUENCY

If $n = 1$ then

$$\mathrm{Fctvn} = \frac{9}{8} - \frac{1}{8M}$$

Else
  Evaluate r(p) and Fac from the recipes below.

$$\mathrm{Fctvn} = \mathrm{Fac} + \frac{1}{6n^2 p}\left[1 - \frac{r^2(p)}{r^2(0)}\right]$$

Endif

$$\mathrm{d.f.} = \frac{p}{\mathrm{Fctvn}}$$

<u>r(p) Recipe</u>

If $p < 1$ then
  $r(p) = 4 - 6p^2 + 3p^3$
Else if $p < 2$ then
  $r(p) = (2 - p)^3$
Else
  $r(p) = 0$
Endif

<u>Fac Recipe</u>

If $p < 1$ then

$$\mathrm{Fac} = p\left(1 - \frac{1}{2}p^2 + \frac{3}{20}p^3 + \frac{3}{20}p^4 - \frac{3}{28}p^5 + \frac{9}{448}p^6\right)$$

Else if $p < 2$ then

$$\mathrm{Fac} = \frac{302 - 103/p}{280} + \frac{(2 - p)^8}{448p}$$

Else

$$\mathrm{Fac} = \frac{302 - 103/p}{280}$$

Endif

Fig. 2. Pseudocode for d.f. recipes. Comments are prefixed by exclamation points.

Some remarks on the limitations of this approach to the assignment of Allan variance confidence intervals are appropriate. The above computations apply only to pure power-law noises. Many phase noise records are better fitted by an additive mixture of two or more of these. The estimated log–log Allan deviation σ-τ curve often shows a transition between two straight-line regions, one for small τ and one for large τ. There is no guide on how to proceed near the crossover. Moreover, one's estimate of the slope of the large-τ part of the curve (for τ greater than 1/20th of the data duration, say) is often unreliable because of the large variance of the estimates $\hat{\sigma}_y^2(\tau)$, heavy correlations between values of $\hat{\sigma}_y^2(\tau)$ for nearby values of τ, and biases from drift or drift removal. The choice of a model on which to base confidence intervals is often unclear.

TABLE I
DEGREES OF FREEDOM OF THE MAXIMAL ESTIMATOR OF $\sigma_y^2(\tau)$, WHERE N = NUMBER OF PHASE SAMPLES SPACED BY τ_0, $n = \tau/\tau_0$, $M = N - 2n$ = NUMBER OF SAMPLES OF SECOND τ-INCREMENT OF PHASE; FOR FLICKER PHASE, $2\pi f_h \tau_0 = 10$

N	n	M	White Phase	Flicker Phase	White Frequency	Flicker Frequency	Random Walk Frequency
9	1	7	3.885	4.180	4.900	6.315	6.323
	2	5	3.237	3.370	3.448	3.347	2.637
	3	3	3.000	2.845	2.250	1.750	1.369
	4	1	1.000	1.000	1.000	1.000	1.000
129	1	127	65.580	71.157	84.889	112.001	112.988
	2	125	64.819	68.586	71.922	71.510	58.229
	4	121	63.305	59.174	42.763	35.865	28.357
	8	113	60.310	45.064	21.535	17.048	13.418
	16	97	54.510	29.844	9.860	7.654	5.955
	32	65	44.762	16.766	4.036	3.039	2.263
	36	57	42.938	15.069	3.461	2.536	1.871
	46	37	37.000	11.396	2.280	1.579	1.271
	56	17	17.000	5.612	1.366	1.101	1.042
	64	1	1.000	1.000	1.000	1.000	1.000
1025	1	1023	526.379	571.378	682.222	901.150	909.432
	2	1021	525.615	556.432	583.919	581.004	473.592
	4	1017	524.089	490.132	354.406	297.572	236.036
	8	1009	521.039	389.458	186.293	147.890	117.252
	16	993	514.953	281.917	93.392	73.048	57.859
	32	961	502.840	187.972	45.753	35.628	28.163
	64	897	478.886	115.944	21.794	16.925	13.319
	128	769	432.510	65.269	9.829	7.592	5.905
	256	513	354.914	32.524	4.005	3.010	2.239
	290	445	339.795	28.586	3.404	2.481	1.829
	370	285	285.000	20.534	2.211	1.539	1.250
	450	125	125.000	9.780	1.331	1.086	1.036
	512	1	1.000	1.000	1.000	1.000	1.000

Percival [16] has argued for the use of direct spectral estimation on phase-noise time series. If the spectral density of phase $S_\phi(f)$ is estimated, then nonparametric estimates both of $\sigma_y^2(\tau)$ and of the variance of conventional time-domain estimators of $\sigma_y^2(\tau)$ can be computed from the spectral estimate $\hat{S}_\phi(f)$. Properties of such estimators remain to be investigated. Moreover, one must take care to assure that estimates of spectral density for low Fourier frequencies f do not suffer from all the same difficulties as time-domain estimates of Allan variance for large τ.

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