A Derivation of the Long-Term Degradation of a Pulsed Atomic Frequency Standard from a Control-Loop Model

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***Abstract*—The phase of a frequency standard that uses periodic interrogation and control of a local oscillator (LO) is degraded by a long-term random-walk component induced by downconversion of LO noise into the loop passband. The Dick formula for the noise level of this degradation is derived from an explicit solution of an LO control-loop model.**

I. Introduction

In 1987, following a suggestion of L. Cutler, Dick [1] described a source of long-term instability for a class of passive frequency standards that includes ion traps and atomic fountains. In these standards, the frequency of a local oscillator (LO) is controlled by a feedback loop whose detection and control operations are periodic with some period T_c. For each cycle, the output of the detector is a weighted average of the LO frequency error over the cycle. The weighting function $g(t)$, derived from quantum-mechanical calculations not addressed here, depends on the method by which the atoms are interrogated by the radio-frequency field generated by upconversion of the LO signal to the atomic transition frequency [1]–[4]. In general, $g(t)$ can be zero over a considerable portion of the cycle. The level of the LO control signal over a cycle is a function of the detector outputs from previous cycles.

A frequency-control loop works by attenuating the frequency fluctuations of the LO inside the loop passband (long-term fluctuations), while tolerating them outside the passband (short-term fluctuations). As Dick saw, though, the periodic interrogation causes out-of-band LO noise power, near the cycle frequency $f_c = 1/T_c$ and its harmonics, to be downconverted into the loop passband, thus injecting random false information about the current average LO frequency into the control signal. This random false frequency correction causes a component of white frequency modulation (FM), or random walk of phase, to persist in the output of the locked LO (LLO) over the long term. Dick gave the formula

$$S_y(0) = 2\sum_{k=1}^{\infty} \frac{g_k^2}{g_0^2} S_y^{\mathrm{LO}}(kf_c) \tag{1}$$

for the white-FM noise level contributed by this effect. Here, $S_y^{\mathrm{LO}}(f)$ is the spectral density of the normalized frequency departure of the free-running LO, and g_k is the Fourier coefficient

$$g_k = \frac{1}{T_c}\int_0^{T_c} g(t)\cos(2\pi k f_c t)\,dt, \tag{2}$$

where $g(t)$ is assumed to be symmetric about $T_c/2$. This level of white FM near Fourier frequency zero contributes an asymptotic component of Allan variance given by

$$\sigma_y^2(\tau) \sim \frac{S_y(0)}{2\tau} \qquad (f_c\tau \to \infty).$$

The purpose of the present paper is to supplement previous derivations [1]–[3], [5] of the Dick formula (1) by an approach that uses an explicit time-domain solution of a simple LO control loop model with a general detection weighting function $g(t)$. Careful interpretation of this solution yields a formula for the LLO frequency spectrum, and conditions for the validity of the Dick formula. The model treated below is not intended as a realistic representation of an actual frequency standard; the goal is to improve understanding of the Dick effect by exhibiting its presence in a model with minimal features. Similar models have been treated by Audoin *et al.* [6], who use an equivalent time-domain solution method, and by Lo Presti *et al.* [5], [7], who use a Fourier transform method. The Lo Presti model also has been treated by the time-domain method [8].

II. Control-Loop Model

Fig. 1 shows the chosen model for an LO control loop, containing both analog and digital elements. All signals are scaled as normalized frequency departure from the ideal frequency determined by the atomic transition. The frequency noise contributed by the free-running local oscillator is $y_{\mathrm{LO}}(t)$. The output LLO frequency is $y(t)$. The error signal

$$\frac{1}{T_c g_0}\int_0^{T_c} g(u)\,y((n-1)T_c + u)\,du \tag{3}$$

from the interrogation of $y(t)$ during the nth cycle $(n-1)T_c < t < nT_c$ is implemented in Fig. 1 by a linear time-invariant filter G with the normalized time-reversed

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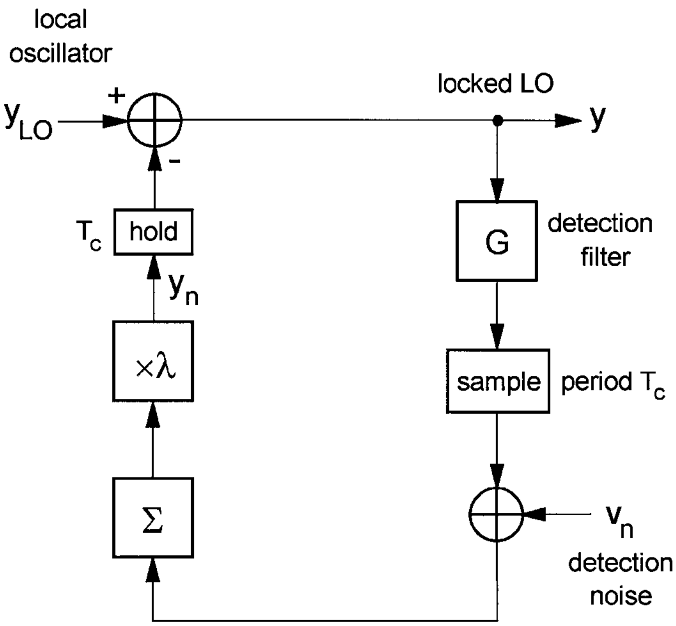

Fig. 1. A feedback-loop model for a local oscillator with periodic interrogation and control. The impulse response of the filter G is a normalized, time-reversed version of the interrogation sensitivity function $g(t)$.

impulse-response function

$$g_1^-(t) = \frac{g(T_c - t)}{T_c g_0}, \quad 0 < t < T_c$$
$$= 0 \quad \text{otherwise}$$

and transfer function

$$G(f) = \int_0^{T_c} g_1^-(t)\, e^{-i2\pi f t}\, dt.$$

The symmetry assumption about $g(t)$ has been dropped. The filter output

$$Gy(t) = \int_0^{T_c} g_1^-(u)\, y(t-u)\, du,$$

sampled at time $t = nT_c$, is exactly (3). The detection noise term v_n can represent photon-count fluctuations in frequency standards with optical detection, for example. The cumulative sum of the error signals, multiplied by a gain factor λ, is the frequency correction y_n, which is applied to the LO during the next cycle $nT_c < t < (n+1)T_c$. Except for initial conditions, the following two equations define the closed-loop model completely:

$$y_n = y_{n-1} + \lambda\left(Gy(nT_c) + v_n\right), \tag{4}$$
$$y(t) = y_{\mathrm{LO}}(t) - y_{n-1}, \quad (n-1)T_c < t < nT_c, \tag{5}$$

in which it is convenient to suppose that n runs through all integers. This system has two inputs, $y_{\mathrm{LO}}(t)$ and v_n, and one output, $y(t)$.

III. The LLO Frequency

The mixed analog-digital system (4), (5) can be solved by eliminating $y(t)$ to get an equation in y_n alone. From (5) we have $Gy(nT_c) = Gy_{\mathrm{LO}}(nT_c) - y_{n-1}$, which, substituted into (4), gives the first-order difference equation:

$$y_n = (1-\lambda)\, y_{n-1} + \lambda w_n, \tag{6}$$

where

$$w_n = Gy_{\mathrm{LO}}(nT_c) + v_n. \tag{7}$$

Assume $0 < \lambda < 1$. Then the general solution of (6) is

$$y_n = \sum_{j=0}^{\infty} \lambda (1-\lambda)^j\, w_{n-j} + C(1-\lambda)^n. \tag{8}$$

From now on we shall ignore the transient part of this solution by setting $C = 0$.

Let us express y_n directly as a function of the inputs $y_{\mathrm{LO}}(t)$ and v_n. Define the discrete-time lowpass filter H_d with weights

$$h_j = \lambda(1-\lambda)^j, \quad j \geq 0,$$

which sum to 1, and transfer function

$$H_d(z) = \sum_{j=0}^{\infty} h_j z^{-j} = \frac{\lambda}{1-(1-\lambda)z^{-1}}, \tag{9}$$

where, from now on, $z = e^{i2\pi f T_c}$. Substituting (7) into (8) gives

$$y_n = \int_0^{\infty} h_c(t)\, y_{\mathrm{LO}}(nT_c - t)\, dt + H_d v_n,$$
$$= H_c y_{\mathrm{LO}}(nT_c) + H_d v_n, \tag{10}$$

in which we have introduced a causal continuous-time filter H_c with impulse response

$$h_c(t) = \sum_{j=0}^{\infty} h_j g_1^-(t - jT_c)$$

consisting of repetitions of $g_1^-(t)$ with exponentially decreasing amplitudes. Notice that $\int_0^{\infty} h_c(t)\, dt = 1$. Its transfer function

$$H_c(f) = \int_0^{\infty} h_c(t)\, e^{-i2\pi f t} dt$$

satisfies

$$H_c(f) = H_d(z)\, G(f). \tag{11}$$

Substituting (10) into (5) gives an explicit piecewise solution for the LLO frequency:

$$y(t) = y_{\mathrm{LO}}(t) - H_c y_{\mathrm{LO}}((n-1)T_c) - H_d v_{n-1},$$
$$(n-1)T_c < t < nT_c. \tag{12}$$

IV. The LLO Frequency Spectrum

Although (12) gives an explicit formula for the output frequency, its interpretation requires careful handling. Under reasonable assumptions (see below) on $y_{\mathrm{LO}}(t)$ and v_n as random processes, we cannot expect the piecewise-defined process $y(t)$ to be stationary, or even to have stationary nth increments for some n. Thus, the author does not know how to assign a spectral density to it. To get around this problem, it is convenient to study the samples $x(nT_c)$ of the LLO time residual $x(t) = \int y(t)\,dt$. The properties of these samples are determined in turn by the properties of the average LLO frequencies

$$\begin{aligned} Ay(nT_c) &= \frac{1}{T_c}\int_{(n-1)T_c}^{nT_c} y(t)\,dt \\ &= \frac{x(nT_c) - x((n-1)T_c)}{T_c} \end{aligned}$$

where A is the moving-average filter whose action on a signal $\xi(t)$ is

$$A\xi(t) = \frac{1}{T_c}\int_0^{T_c} \xi(t-u)\,du.$$

Its transfer function is

$$A(f) = \frac{1 - z^{-1}}{i2\pi f T_c}.$$

Applying A to (12) gives

$$\begin{aligned} Ay(nT_c) &= Ay_{\mathrm{LO}}(nT_c) \\ &\quad - H_c y_{\mathrm{LO}}((n-1)T_c) - H_d v_{n-1}. \end{aligned} \tag{13}$$

We are now going to derive the spectrum of the discrete-time process $Ay(nT_c)$ defined by (13). To this end, consider the auxiliary process defined by

$$Y(t) = Ay_{\mathrm{LO}}(t) - H_c y_{\mathrm{LO}}(t - T_c),$$

which is obtained from $y_{\mathrm{LO}}(t)$ by a linear time-invariant filter B with transfer function

$$B(f) = A(f) - z^{-1} H_d(f)\,G(f). \tag{14}$$

Assume that $y_{\mathrm{LO}}(t)$ is a mean-continuous random process with stationary first increments and a two-sided (even) spectral density $S_y^{\mathrm{LO}}(f)$, which necessarily satisfies

$$\int_0^{f_c} S_y^{\mathrm{LO}}(f)\,f^2 df < \infty, \quad \int_{f_c}^{\infty} S_y^{\mathrm{LO}}(f)\,df < \infty \tag{15}$$

[9]. The first condition in (15) allows any power law spectrum $|f|^{\alpha}$ for $\alpha > -3$; linear combinations of such spectra constitute the spectra that are customarily attributed to oscillators. For $\alpha \geq -1$, the second condition in (15) requires a high-frequency rolloff of the $|f|^{\alpha}$ behavior.

The assumption (15) makes the process $Y(t)$ stationary: because $A(f)$, $H_d(z)$, and $G(f)$ are all $1 + O(f)$ as $f \to 0$, we see from (14) that $B(f) = O(f)$. Thus $|B(f)|^2$ attenuates any low-frequency divergence of $S_y^{\mathrm{LO}}(f)$ allowed by (15), leaving an integrable two-sided spectral density

$$S_Y(f) = \left|A(f) - z^{-1} H_d(z)\,G(f)\right|^2 S_y^{\mathrm{LO}}(f). \tag{16}$$

The first two terms of the right side of (13) are just the samples $Y(nT_c)$, which constitute a discrete-time stationary process whose two-sided spectral density is

$$\sum_{k=-\infty}^{\infty} S_Y(f + kf_c), \quad |f| \leq f_c/2.$$

The terms with $k \neq 0$ account for the Dick effect. Let the detection noise process v_n be independent of $y_{\mathrm{LO}}(t)$ and stationary, with two-sided spectral density $S_v(f)$. Then the process $Ay(nT_c)$ given by (13) is a stationary discrete-time process with two-sided spectral density

$$S_{Ay}(f) = S_{Ay}^0(f) + S_{Ay}^1(f), \quad |f| \leq f_c/2,$$

where

$$\begin{aligned} S_{Ay}^0(f) &= \left|A(f) - z^{-1} H_d(z)\,G(f)\right|^2 S_y^{\mathrm{LO}}(f) \\ &\quad + |H_d(z)|^2 S_v(f), \end{aligned} \tag{17}$$

the main part, so to speak, and

$$\begin{aligned} S_{Ay}^1(f) &= \sum_{k \neq 0} \left| \frac{1 - z^{-1}}{i2\pi(fT_c + k)} - z^{-1} H_d(z)\,G(f + kf_c) \right|^2 \\ &\quad \times S_y^{\mathrm{LO}}(f + kf_c), \end{aligned} \tag{18}$$

the aliased part, where the sum includes both positive and negative k.

An example of these frequency spectra is shown in Fig. 2, in which $T_c = 1$ s, $S_y^{\mathrm{LO}}(f) = |f|^{-1}$ (flicker FM)[1], $g(t) = 1$ for $T_c/2 < t < T_c$ and 0 otherwise, and $\lambda = 1/10$. Detection noise is omitted. The spectra are plotted up to frequency $f_c/2$. Harmonics through order 5 are used to approximate the series in (18). Despite the attenuation of the main part of the LLO spectrum from the LO spectrum below the loop bandwidth, the white-FM contribution of the aliased part is dominant only for frequencies below $10^{-4} f_c$. The bandwidth of the aliased white-FM noise is approximately the same as the loop bandwidth.

V. The Dick Formula

In general, the aliased part (18) of the LLO frequency spectrum introduces a long-term white-FM spectral component if (18) is continuous and positive at $f = 0$. Reasonable mathematical conditions on the weighting function and LO frequency spectrum will guarantee the continuity

[1] Violation of the second condition in (15) does not really matter.

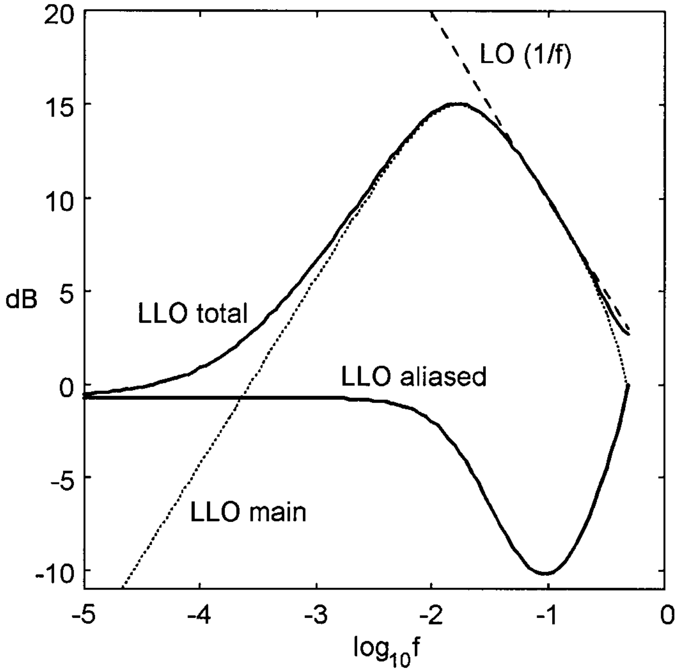

Fig. 2. Frequency spectra for a simple example with flicker-FM local-oscillator noise and a 1-second cycle. The aliased spectrum at low frequencies is responsible for the Dick effect.

of the aliased spectrum. For example, if $g(t)$ is square-integrable for $0 < t < T_c$, and $S_y^{\mathrm{LO}}(f)$ is continuous and bounded for $|f| \geq f_c/2$, then it can be proved that the right side of (18) is a uniformly convergent series of continuous functions on $|f| \leq f_c/2$. Consequently, the sum of the series is a continuous function. Letting $f = 0$ in (18) gives

$$S_{Ay}^{1}(0) = 2\sum_{k=1}^{\infty} \left| G(kf_c)^2 \right| S_y^{\mathrm{LO}}(kf_c). \tag{19}$$

For the example of Fig. 2, $S_{Ay}^{1}(0) = \left(8/\pi^2\right) \sum_{j=0}^{\infty} (2j + 1)^{-3} = 0.853$. The formula (19) holds for one-sided spectral densities also. Finally, if $g(t)$ is symmetric about $T_c/2$, then $G(kf_c) = g_k/g_0$. Thus (19) reduces to the Dick formula (1).

VI. Remarks

The Dick effect can be hidden by the main part (17) of the LLO spectrum. If the detection noise v_n is white, then the term $|H_d(z)|^2 S_v(f)$ competes directly with the Dick effect as another white-FM noise at low frequencies. The basic action of the control loop operates on the LO frequency by a filter with frequency response $\left|A(f) - z^{-1}H_c(z)\right|^2$, which, as we observed, is $O\left(f^2\right)$ as $f \to 0$. Thus, the filter adds 2 to the exponent of any low-frequency power law that $S_y^{\mathrm{LO}}(f)$ obeys. If $S_y^{\mathrm{LO}}(f)$ is more divergent than f^{-2} (random walk FM), then $S_{Ay}^{0}(f)$ is unbounded near $f = 0$, hence masks the Dick effect. Random walk FM in the LO is transformed to another white FM component in the LLO. Anything less divergent, like flicker FM, is transformed to an LLO spectral density that tends to zero at low frequencies. In this case, the Dick effect and the detection noise predominate in the long term.

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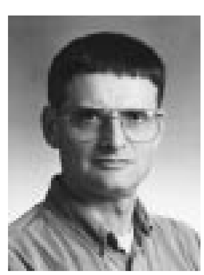

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