$$F_2(\hat{a}) = E = \begin{bmatrix} G_b \\ G_c \end{bmatrix} E \left\{ \begin{bmatrix} \dfrac{x(n)}{\prod_{l=0}^{N-1} (1 - \hat{a}_l^* q^{-1})} \\ \vdots \\ \dfrac{x(n - 2N + 1)}{\prod_{l=0}^{N-1} (1 - \hat{a}_l^* q^{-1})} \end{bmatrix} \begin{bmatrix} \dfrac{x(n)}{\prod_{l=0}^{N-1} (1 - \hat{a}_l q^{-1})}, \cdots, \dfrac{x(n - 2N + 1)}{\prod_{l=0}^{N-1} (1 - \hat{a}_l q^{-1})} \end{bmatrix} \right\} \begin{bmatrix} G_b \\ G_c \end{bmatrix}^H \tag{A7}$$

where the ith row of G_b corresponds to the coefficients of the polynomial $\frac{1}{2}D_i(z^{-1}) = \prod_{k=0, k \neq i}^{N-1}(1 - W_N^{k+i_0} z^{-1})(1 - \hat{a}_k^* z^{-1})$, and the ith row of G_c corresponds to the coefficients of the polynomial $\frac{1}{2}z^{-1}D_i(z^{-1})$. Thus

$$\begin{bmatrix} G_b \\ G_c \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \nabla_{\hat{b}}\hat{\zeta} \\ \nabla_{\hat{c}}\hat{\zeta} \end{bmatrix} \tag{A8}$$

and its determinant is given by (5) which is nonzero if $\hat{a}_i \neq \hat{a}_k$ for all $i \neq k$. Since $\prod_{l=0}^{N-1} (1 - \hat{a}_l^* q^{-1}) = 1 + \sum_{l=1}^{N} \hat{\xi}_l q^{-l}$ which is a stable polynomial with real coefficients, by [14, lemma 4.7], the matrix in the middle is nonsingular if $\{x(n)\}$ is persistently exciting of order $2N$. This completes the proof.

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Orthogonal Sets of Data Windows Constructed from Trigonometric Polynomials

CHARLES A. GREENHALL

Abstract—**This correspondence gives suboptimal, easily computable substitutes for the discrete prolate spheroidal windows used by Thomson for spectral estimation. Trigonometric coefficients and energy leakages of the window polynomials are tabulated.**

I. Motivation

In the Thomson method [1] for estimating the power spectrum of a stationary time series from N values $x[0], \cdots, x[N-1]$, a frequency band $[f_0 - W, f_0 + W]$ is chosen, and an estimate for the spectral density $S(f_0)$ is computed as an average of windowed periodogram values, namely

$$\hat{S}(f_0) = \frac{1}{K}\sum_{k=0}^{K-1} \left| y_k(f_0) \right|^2 \tag{1}$$

where

$$y_k(f) = \sum_{n=0}^{N-1} x[n]\, v_k[n; N, W]\, e^{-i2\pi f n}. \tag{2}$$

The window sequences $v_0, \cdots, v_{N-1}$ are the discrete prolate spheroidal sequences (DPSS) of Slepian [2]. They are orthonormal and are *leakage optimal* over the space of sequences index-limited to $0, \cdots, N-1$, in the sense that 1) v_0 has the smallest leakage of all nonzero elements; 2) for $k > 0$, v_k has the smallest leakage of all nonzero elements orthogonal to $v_0, \cdots, v_{k-1}$. In this context, the *leakage* $L(g, W)$ of a function g of discrete or continuous time is defined as the fraction of its total energy contained in frequencies outside $[-W, W]$. The leakage $L(v_k, W)$ increases with k and decreases with W. By virtue of the orthogonality of the v_k, the estimate (1) has approximately $2K$ degrees of freedom if x is Gaussian, $S(f)$ is nearly constant for $|f - f_0| \leq W$, and the leakage of v_{K-1} is small. Thus, by adjusting W and K, one can achieve a tradeoff in (1) among variance, resolution, and the influence of frequencies outside $[f_0 - W, f_0 + W]$.

Since the DPSS are somewhat difficult to compute, the design of easily computable suboptimal substitutes for them may be of value. In view of Nuttall's constructions of windows from cosine polynomials of low degree [3], one might expect trigonometric polynomials with both sines and cosines to make attractive materials for construction of DPSS substitutes. In fact, this idea has already

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been realized by Bronez [4, p. 1869] in his recent extension of the Thomson method to the more general situation of unevenly spaced and multidimensional data. The coefficients of his polynomials and their leakages are, respectively, the eigenvectors and eigenvalues of a certain matrix whose coefficients depend on N, the number of data. The aim of this note is to simplify the situation further for evenly spaced one-dimensional data by deriving the coefficients and leakages of an orthonormal set of continuous-time trigonometric polynomials that do not depend on N. They are converted to discrete-time data windows by sampling them at N properly chosen points.

II. Continuous-Time Windows

We shall always use w to denote bandwidth in terms of the fundamental frequency unit, which is $1/N$ for windows on $0, \cdots, N-1$, and 1 for windows on $(-1/2, 1/2)$, as constructed below. It will be assumed that w is an integer (for the author's convenience only). Consider a time-limited trigonometric polynomial

$$\phi(x) = \sum_{\nu=-M}^{M} c[\nu]\, e^{i2\pi\nu x}, \qquad |x| < 1/2$$
$$= 0, \qquad |x| \geq 1/2$$

of degree $\leq M$. Its Fourier transform is

$$\Phi(y) = \sum_{\nu=-M}^{M} c[\nu]\, s(y-\nu) \tag{3}$$

where

$$s(y) = \frac{\sin \pi y}{\pi y}. \tag{4}$$

The polynomials we seek can be defined immediately: their coefficient arrays are normalized eigenvectors of the positive-definite matrix

$$A[i, j] = \int_{|y|>w} s(y-i)\, s(y-j)\, dy, \qquad i, j = -M \text{ to } M$$

and their leakages are the eigenvalues. Denote the resulting polynomials by $\phi_k(x; w, M)$, $k = 0$ to $2M$, and their coefficients by $c_k[\nu]$, $\nu = -M$ to M, where the leakages $L(\phi_k, w)$ are taken in increasing order. These polynomials, which are orthonormal and leakage optimal over the space of polynomials of degree less than or equal to M, will be called *trig prolates*, because they can be regarded as finite-dimensional analogs of the prolate spheroidal wave functions (PSWF) (Slepian [5]). The symmetry of A about its reverse diagonal forces the eigenfunctions to be either even or odd (the odd ones are multiplied by $\pm i$ to make them real), and we find empirically that the above indexing gives ϕ_k the parity of k. We remark that the trig prolates share with the PSWF the property of double orthogonality: their transforms $\Phi_k(y; w, M)$ are orthogonal over $[-w, w]$ as well as $(-\infty, \infty)$.

The entries of A were computed as linear combinations of the integrals

$$\int_0^1 \left| \frac{\sin \pi y}{\pi(n+y)} \right|^2 dy, \qquad \int_0^1 \left| \frac{\sin^2 \pi y}{\pi^2(n+y)} \right| dy$$
$$n = 0 \text{ to } w + M - 1$$

which were computed by Romberg quadrature. The eigenvalues and eigenvectors were computed by EISPACK routines [6], [7]. Although the leakages decrease if M increases, setting $M = w$ gives adequate performance, as shown below.

Table I gives the coefficients and leakages of the trig prolates for $w = 2$ to 5, $M = w$, and for all k such that $L(\phi_k, w) < 0.01$. Fig. 1 shows the frequency response $|\Phi_k(y; 4, 4)|^2$ for $k = 0$ and 4. Comparing these with Thomson's graphs of the frequency responses of the DPSS for large N [1, fig. 2], we see that the maximum sidelobes of the trig prolates are at most 2.5 dB above those

TABLE I

Sine-Cosine Coefficients $a_k[\nu]$ and Leakage $L(\phi_k, w)$ for Trig Prolate $\phi_k(x; w, w) = a_k[0] + 2\sum_{\nu=1}^{M} a_k[\nu]\cos 2\pi\nu x$ (k Even), or $2\sum_{\nu=1}^{M} a_k[\nu]\sin 2\pi\nu x$ (k Odd)

w = M = 2

k	0	1
L	.8901E-04	.3254E-02
ν	$a_k[\nu]$	
0	.8202108	0.0
1	.4041691	.7007932
2	.0165649	.0942808

w = M = 3

k	0	1	2	3
L	.2113E-06	.1520E-04	.4394E-03	.6810E-02
ν	$a_k[\nu]$			
0	.7499700	0.0	.4969513	0.0
1	.4596063	.6507499	-.3050683	.2731233
2	.0867984	.2765560	-.5312499	-.6397174
3	.0007513	.0064282	-.0350227	-.1271430

w = M = 4

k	0	1	2	3	4
L	.4376E-09	.4203E-07	.1965E-05	.5382E-04	.9029E-03
ν	$a_k[\nu]$				
0	.6996910	0.0	.4783016	0.0	-.3862293
1	.4830013	.5927723	-.1666510	.3540569	.3223025
2	.1473918	.3805986	-.5724443	-.4929565	-.0856254
3	.0141997	.0613650	-.1736202	-.3626279	-.5584413
4	.0000368	.0003329	-.0022015	-.0117722	-.0484379

w = M = 5

k	0	1	2	3	4	5	6
L	.1056E-11	.1148E-09	.6265E-08	.2307E-06	.6065E-05	.1112E-03	.1411E-02
ν	$a_k[\nu]$						
0	.6632850	0.0	.4560698	0.0	-.3821638	0.0	.3246026
1	.4915713	.5401300	-.0704481	.3866087	.2527019	-.2216043	-.2957322
2	.1927963	.4383060	-.5519198	-.3363930	.1138304	.3885522	.1964585
3	.0347859	.1266343	-.2915206	-.4760267	-.5457777	-.3657298	.0266965
4	.0019243	.0105462	-.0379143	-.1037856	-.2286313	-.4072901	-.5631039
5	.0000018	.0000191	-.0001319	-.0007467	-.0037712	-.0165910	-.0588589

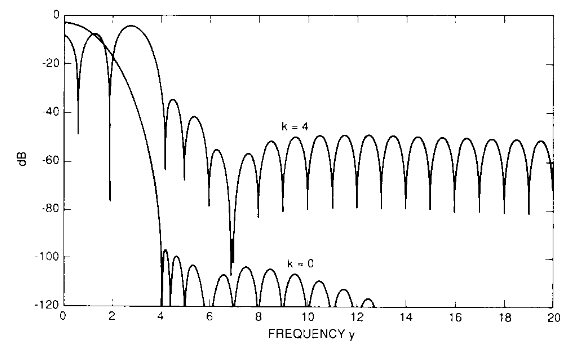

Fig. 1. Frequency response of trig prolate windows for bandwidth $w = 4$, degree $M = 4$. The total energy for each window is 1.

of the corresponding DPSS, although the sidelobe structure of the trig prolates is less regular.

III. Discrete-Time Windows

An orthogonal set of windows for data on $0, \cdots, N-1$ and bandwidth $W = w/N$ is constructed by sampling the ϕ_k as follows:

$$u_k[n; N, W, M] = \phi_k\left(\frac{n - (N-1)/2}{N}; w, M\right), \qquad n = 0 \text{ to } N-1,$$
$$k = 0 \text{ to } 2M. \tag{5}$$

Notice that the denominator is N instead of $N - 1$. This has two beneficial effects: 1) the basis functions $e^{i2\pi\nu x}$ remain orthogonal when so sampled; 2) their discrete-time transforms are more closely related to their continuous-time transforms (see below).

The discrete-time windows u_k will be called *sampled trig prolates*. Orthogonality is preserved, namely, $\sum_{n=0}^{N-1} u_i[n]u_j[n] = N\delta_{ij}$. Their discrete-time Fourier transforms are

$$U_k(f; N, W, M) = e^{-i\pi(N-1)f} \sum_{\nu=-M}^{M} c_k[\nu]\, s(Nf - \nu; N) \tag{6}$$

TABLE II
RATIO (DECIBELS) OF SAMPLED TRIG PROLATE LEAKAGE TO OPTIMAL DPSS LEAKAGE. THE FIRST ENTRY IS FOR $N = 8w$, THE SECOND FOR $N = 16w$

k / w	0	1	2	3	4	5	6
2	2.1, 1.9	1.2, 1.2					
3	2.7, 2.0	2.4, 2.2	2.0, 1.9	1.3, 1.3			
4	3.7, 1.8	2.8, 1.7	2.6, 2.1	2.4, 2.2	1.9, 1.9		
5	5.4, 2.6	4.2, 2.1	3.3, 1.9	2.7, 2.0	2.6, 2.2	2.4, 2.2	1.9, 1.8

where

$$s(y; N) = \frac{\sin \pi y}{\sin(\pi y/N)} \tag{7}$$

(compare with (3) and (4)). A spectral estimate of Thomson type is obtained by using $(1/\sqrt{N})u_k[n; N, W, M]$ in place of $v_k[n; N, W]$ in (1) and (2).

IV. COMPARISON WITH OPTIMAL WINDOWS

How much leakage performance is lost by the use of these suboptimal windows? Let $L(\phi_k, w)$, $L(u_k, N, W)$, and $1 - \lambda_k(N, W)$, where $NW = w = M$, be, respectively, the leakages of the trig prolates, the sampled trig prolates, and the optimal DPSS. Evaluating $L(u_k)$ by means of the quadratic form in the numerator of (32) of Bronez's paper [3], we find that $L(u_k)$ is between 0 dB and 1.2 dB less than $L(\phi_k)$ for the instances of w and k given in Table I and for $N = 8w$. For $N = 16w$, replace 1.2 dB by 0.6 dB. Thus, the sampled trig prolates have slightly *less* leakage than the trig prolates. Table II gives the ratio of sampled trig prolate leakage to DPSS leakage, which was computed by solving the eigensystem given by Thomson [1, eq. (2.9)]. For $N = 8w$, the leakages of the sampled trig prolates are 1.2 to 5.4 dB greater than those of the optimal DPSS; for $N = 16w$, the range is 1.2 to 2.6 dB. The leakages of the corresponding Bronez discrete polynomial windows, which form the leakage-optimal set of discrete-time trigonometric polynomials of degree less than or equal to M, necessarily lie between those of the sampled trig prolates and those of the DPSS.

V. CONCLUSIONS

We have described several orthonormal systems of data windows, the sampled trig prolates, which can be used in the Thomson method of spectral estimation. For $w = NW = 2$ to 5, and $8W$ not greater than the Nyquist frequency (i.e., $N \geq 16w$), the user of these windows pays a leakage penalty at most 2.6 dB for not using the optimal DPSS windows. In return, one has only to evaluate the trigonometric polynomials of degree w from Table I at N points according to (5). In contrast, the evaluation of the DPSS windows requires the solution of an $N \times N$ symmetric Toeplitz matrix eigensystem. If N is large, one can proceed by solving a symmetric $J \times J$ eigensystem obtained from the approximation of a certain integral operator by Gaussian quadrature, in which the required number of knots J depends on the details of floating-point hardware and mathematical software [1, pp. 1090–1091]. The prospective user of the Thomson method might regard the 2.6-dB penalty as an acceptable price for avoiding these complexities.

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AR Model Identification With Unknown Process Order

S. K. KATSIKAS, S. D. LIKOTHANASSIS, AND D. G. LAINIOTIS

***Abstract*—A new method for simultaneously selecting the order and identifying the parameters of an AR model has been developed. The consistency of the method has been proved for general (not necessarily Gaussian) data pdf's. As a by-product of the derivation of the consistency of our method, the consistency of the PLS criterion is proven elegantly and concisely. Simulation experiments indicate that the proposed method is 100% successful in selecting the correct model order and that it accurately identifies the model parameters. Furthermore, it does so in very few steps. The algorithm can be parallely implemented and also a VLSI implementation is feasible.**

I. INTRODUCTION

The problem of fitting an autoregressive (AR) model to a given time series is a fundamental one in linear prediction, system identification, and spectral analysis. Furthermore, it arises in a large variety of applications, such as adaptive control, speech analysis and synthesis, channel equalization, EEG and ECG analysis, geophysical data processing, etc.

The problem can be formally described as follows: given a set of samples from a discrete time process $\{y(k), 0 \leq k \leq N-1\}$, it is desired to obtain the set of coefficients $\{a_i\}$ which yields the best linear prediction of $y(N)$ based on all past samples:

$$\hat{y}(N/N-1) = \sum_{i=1}^{\theta} a_i y(N-i) \tag{1}$$

where $\hat{y}(N/N-1)$ denotes the predicted value of $y(N)$ based on measurements up to and including $y(N-1)$. θ is the *order* of the predictor and the a_i's are the *predictor coefficients*.

Clearly, the problem is twofold: one has both to select the order of the predictor and to compute the predictor coefficients. Perhaps the most crucial part of the problem is the former. Many methods exist for AR model order selection, the most well known and widely used among them being the ones proposed by Akaike [1], [2], Parzen [3], Rissanen [4], Hannan and Quinn [5]. All these are based on the assumption that the data are Gaussian and upon asymptotic results. Hence, strictly speaking, their applicability is limited only

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